II. Solution by T. M. Blakslee, Ames, Iowa.

Let the coördinates of A be (0, 0), of B, $(c, 0)$, of C, (h, k), and let the lengths of the sides of the triangle be a, b, c. As the slope of BC is $k/(h - c)$, the theorem is proved if we show that the slope of OQ is $(c - h)/k$.

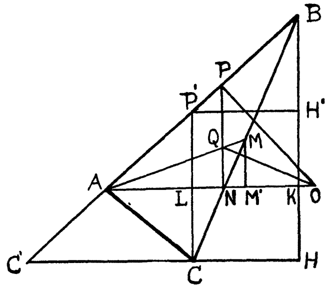
The coördinates of M are $[(c + h)/2, k/2]$, and the equation of AM is $y = kx/(c + h)$. The coördinates of N are $(\lambda b_1, \lambda k)$, where $b_1 = b + h$ and $\lambda = c/(b + c)$, and the equation of AN is $y = kx/b_1$. The equation of NP is $ky + b_1x = \lambda(k^2 + b_1^2)$, and hence the coördinates of P are $(\lambda s/b_1, 0)$ where $s = b_1^2 + k^2$. Since the x-coördinate for P and O are the same, we have for the coördinates of O, $(\lambda s/b_1, \lambda ks/b_1^2)$. The coördinates of Q are found from the equations of AM and NP to be $[\lambda s(c + h)/(k^2 + b_1c + b_1h), \lambda sk/(k^2 + b_1c + b_1h)]$. The slope of OQ is now found to be $[k^2 + b_1c - b_1(b_1 - h)]/(kb_1)$ and, using the relations $b_1 - h = b$ and $k^2 = b_1(b - h)$, this reduces to $(c - h)/k$. The proof for the external bisector may be carried through in the same manner.

III. Solution by the Proposer.

The truth of the theorem is evident if the angles B and C are equal. Suppose then that the angle B is the smaller of the two.

Prolong BA to C' so that $AC' = AC$. Then $C'C$ is parallel to AN. Let BH be the perpendicular from B upon $C'C$ produced to H. Draw CP' parallel to HB, cutting AN in L and AB in P'; also MM' parallel to HB meeting AN produced in M'. Draw $P'H'$ parallel to CH, meeting HB in H'. Then AN produced bisects HH' at K.

Since M is the middle point of BC, $M'M = (KB - KH)/2 = (KB - KH')/2 = H'B/2$ and $AM' = (AL + AK)/2 = C'H/2$. From similar triangles we have



$$\frac{NQ}{AN} = \frac{M'M}{AM'} = \frac{H'B}{C'H}, \quad \text{or} \quad \frac{NQ}{H'B} = \frac{AN}{C'H} = \frac{NP}{HB}.$$

Therefore

$$\frac{NQ}{NP} = \frac{H'B}{HB} = \frac{CH}{C'H}.$$

But

$$\frac{NP}{NO} = \frac{AN}{NP} = \frac{C'H}{HB}.$$

Therefore $NQ/NO = CH/HB$. Hence the right triangles ONQ, CHB are similar, and, since ON is perpendicular to HB, OQ must also be perpendicular to BC.

Also solved by P. J. da Cunha.

2851 [1920, 377]. Proposed by Hillel Poritsky, Cornell University.

Does there exist an analytic function satisfying the functional equation $f(z + 1) = e^{f(z)}$?

Solution by A. A. Bennett, University of Texas.

This is a typical problem of iteration and may be answered at once in the affirmative by reference to the usual methods in that theory.

It is necessary to find the fixed points of the iteration of e^x, that is, solutions of $x = e^x$. There are three such points; two proper finite points[1] ($.318 \pm i\, 1.337$, approximately), and the point at infinity which latter satisfies the relation only for certain methods of approach. Let a denote one of the two finite fixed points above mentioned, then $a = e^a$, and the constant function a, is one solution of the problem. A one-parameter family of non-constant solutions including a as a member is readily found by the use of undetermined coefficients as follows:

Write $g(ax - a^2 + a) = e^{g(x)}$ where $g(ax - a^2 + a) = f(z + 1)$ and $g(x) = f(z)$; that is, $x = a^z + a$, and $g(x) = f[\log_a (x - a)]$. Now $g(x)$ may be expanded in the form,

$$g(x) = a + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots,$$

[1] Compare this Monthly, *1921*, 141–142, footnote.

where c_1 is chosen arbitrarily and the remaining coefficients obtained by comparison in the expansion of the two members of the relation[1]

$$g(ax - a^2 + a) = e^{g(x)}.$$

The series will necessarily converge in accordance with the general theory. Since there are two such values, a, there exist two one-parameter families of analytic functions satisfying the problem.

The above solutions are not real functions, so that the existence of a real analytic solution, other than plus infinity, remains to be examined. A smooth curve can be drawn containing one parameter of translation satisfying the problem, except for its possible non-analytic character. Owing to the highly singular character of the point at infinity, in connection with this problem the usual methods cannot be applied to this point. The following method of approach may prove suggestive. Modify the problem so that the fixed point at infinity shall appear at the origin; thus replace f and z by their reciprocals h and t, giving $h(t/(1+t)) = e^{-1/h(t)}$. Choose the parameter of translation so that $h(1) = 1$. Then $h(1/2) = 1/e$, $h(1/3) = 1/e^e \cdots$. At the points where $h(t)$ is thus determined, it approaches zero with extraordinary rapidity as t moves in toward the origin. Use Newton's or some other interpolation formula to obtain a real function $H_1(t)$ coinciding with $h(t)$ at $t = 1, 1/2, 1/3, \cdots, 1/n, \cdots$ and smooth, in the intermediate intervals. Now H_1 may not satisfy the functional relations but $H_1(t)$ and $-1/\log H_1(t/(1+t))$ both coincide with $h(t)$ at the points $t = 1, 1/2, 1/3, \cdots$. The real solution desired may be expected to lie in most places between them, since if they coincide they will form a solution. Take a mean of $H_1(t)$ and $-1/\log H_1(t/(1+t))$ and call it $H_2(t)$. The particular method of choosing a mean is not significant since only a process of successive approximations is attempted. From $H_2(t)$, form $-1/\log H_2(t/(1+t))$; take $H_3(t)$ as a mean of these and proceed thus indefinitely. Inspection would suggest that the process might be arranged to lead to a determinate real analytic function in the limit, which function $H(t)$ would coincide with $-1/\log H(t/(1+t))$, so that $H(t)$ is a solution, $h(t)$, desired for the modified problem. Thence, the solution, $f(z)$, of the given problem is obtained by taking reciprocals. The above method lends itself to numerical handling, but the proof of the convergence to an analytic function will, of course, involve the usual theoretical complications.

2862 [1920, 428]. Proposed by J. L. RILEY, Stephensville, Texas.

Show that the whole area commanded by a gun on a hillside is an ellipse whose focus is at the gun, whose eccentricity is the sine of the inclination of the hill to the horizon, and whose semi-latus rectum is twice the greatest height to which the gun could send a ball.

SOLUTION BY A. V. RICHARDSON, Bishop's College, Lennoxville, Quebec.

Let G be the position of the gun, AGA' the line of greatest slope, α the inclination of the hillside, and GL the maximum range in a direction making an angle θ with the line of greatest slope.

Also let β be the angle which GL makes with its projection GN on the horizontal plane through G. Then if u, $\phi + \beta$ represent the muzzle velocity and angle of elevation, respectively, for the maximum range GL we have

$$\begin{aligned} u \cos(\phi + \beta) \cdot t &= GN \qquad (t = \text{time of flight}) \\ &= GL \cos\beta. \qquad (1) \end{aligned}$$

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[1] We may use the equation $ag'(ax - a^2 + a) = g'(x)g(ax - a^2 + a)$, g' denoting derivative. We shall find each c equal to a polynomial in the c's with lower suffix; in fact, c_n will be equal to c_1^n times a function of a alone, so that we can put $c_1 = 1$ in these equations. We shall have $2(a-1)c_2 = 1$, and in general

$$(n+1)(a^n - 1)c_{n+1} = a^{n-1}c_n + 2a^{n-2}c_2c_{n-1} + \cdots + (n-1)ac_{n-1}c_2 + nc_n.$$

Now suppose $|c_r| \leqq k^{r-1}$, $r = 2, 3, \cdots n$. Then

$$(n+1)|a^n - 1||c_{n+1}| \leqq [|a|^{n-1} + 2|a|^{n-2} + \cdots + n]k^{n-1} < (n+1)\frac{|a|^n - 1}{|a| - 1}k^{n-1}.$$

Also $|a^n - 1| \geqq |a|^n - 1$, and hence $|c_{n+1}| \leqq k^{n-1}/(|a| - 1)$.

Furthermore, $|c_2| < 1/(|a| - 1)$. Therefore, if we take $k \geqq 1/(|a| - 1)$, we shall have for all values of n $|c_n| \leqq k^{n-1}$.

The radius of convergence of the series for $g(x)$ is at least equal to $1/k|c_1|$ or $(|a| - 1)/|c_1|$. —EDITOR.